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SIMULTANEOUS MEASURES OF HUMAN RELATIONS AND EMOTIONAL ACTIVITY

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The quantitative description of human relations is based upon the measurement of the interaction of individuals. This description involves the continuous measurement of the durations of actions and inactions of observed persons as one acts and another is silent, and also includes all instances of interruptions (when both act) and of failures to respond (when both are silent). The methods of measurement and the discriminations used have been described elsewhere,[1,2] but it must be repeated here that in these measurements no attempt is made to evaluate the subjective experience of the individuals by interpreting the "meaning" of the words and gestures making up the actions. It is believed that a satisfactory description of human relations (the adjustment of individuals) is to be obtained from the quantitative analysis of the measured values of the actions and inactions in interaction in terms of the changes in synchronization of the interacting individuals. Since the actions of organisms, whether "spontaneous" or in response to stimuli, clearly involve modifications of the internal environment, we should expect that the "emotions" which play such a prominent part in our subjective experience of relations with other people would be directly associated with the interaction. Hence, simultaneous measures of human relations and emotional activity ought to provide a test of this hypothesis.

Estimates of the degree of adjustment of individuals to one another and of the associated emotions have always played a large part in the accounts of human behavior by social and psychological investigators. Ordinarily, these accounts have had a frankly subjective character, but in some instances various measures of physiological response have been utilized. These have included (to name only those most frequently used) the measurement of the electrical activity of the brain,[3] the electro-galvanic response,[4] the heart rate,[5] the respiration rate,[6] blood pressure,[6] muscular tension.[7]

In each case, however, the stimulus eliciting the measured response has always been qualitative—the presentation to the subject of selected symbols supposed to have "emotional" significance. Not only is there no precise evidence of the connection of any given symbol with the behavior even of a single individual, but the conditions of experiment, when involving interaction, introduce further difficulties if interaction can be shown to produce alterations in emotional response. In any case, the use of operations to obtain a measurement of the magnitude of a physiological response is liable to conceal the fact that the nature of the stimulus is such that we are not comparing our measured values with measured values (for the stimulus) of known dimensions, but rather with qualitatively discrete phenomena whose only quantification derives from the measured response.

In laboratory studies of emotion by physiologists, where the purpose of the experiment is to describe the changes in the organism associated with the overt manifestation of emotion, there is more evidence, from the nature of the experiments, of the relationship between interaction and emotional activity. In these studies the laboratory animal, usually a cat, is tied to an animal board and then put into a state of rage or fear. This is often done by taking advantage of the previous conditioning of the animal and, for example, bringing in a dog who barks at and tries to attack the cat. The cat responds to the succession of stimuli provided by the dog's actions, and the overt activities of the skeletal muscles of the cat in interaction are associated with changes in the internal environment produced by the activation of the autonomic nervous system, measured by changes in heart rate, blood sugar and adrenalin concentration, and so on.[8,9] These manifestations of emotional activity involving changes in the neuro-humoral system are regarded as the response of the organism to disturbances of its equilibrium, or of what Cannon calls homeostasis.[10] These disturbances may be divided into stimuli of "high" intensity and stimuli which are repeated a number of times. In each of these divisions the factor of the durations of the periods of the stimulus and the frequency of presentation of the stimulus are implicit in the experimental situation, but are ordinarily unspecified.

The use of the electroencephalograph as a measure of emotional activity developed from the observations of Hoagland, *et al.*, on schizophrenics undergoing insulin treatment.[11] In studying the brain wave records marked variations in the so-called delta waves were noticed. It was found that an estimate of the amount of delta activity provided a useful and quantitative account of changes in the patient's condition, not only during hypoglycemia but also from day to day. In a number of cases studied, changes in the state of the patient as reported from the clinical history were correlated with changes in the values of the delta index (the measure contrived by Hoagland[11] to quantify the delta waves). In the course of these

observations, it was also observed that marked increases in the delta index were associated with emotional crises, and a series of observations was then made to test the hypothesis that delta wave activity is directly associated with emotion. The technique used was the so-called psychic probe, in which material taken from the patient's case history is repeated in the form of questions to the patient. These questions referred to situations apparently having strong emotional connotations. Between each question, there was a period of silence. Marked increases in the delta index were correlated with the periods of questioning.[3]

It was then suggested that the principal source of the delta waves was the hypothalamus, since the work of Cannon, Bard and others has demonstrated that the hypothalamus is the controlling mechanism in the neurohumoral system (responsible for emotion).[9,12] Using a method developed by Grinker,[13] simultaneous records of psychic probe observations were made from the occipital region and from a region near the hypothalamus. Not only does the evidence indicate the strong predominance of the delta waves in the hypothalamic record, but Hoagland, *et al.*, were able to show that the delta waves at the hypothalamic lead preceded the corresponding delta waves at the cortical lead by an average of about four milliseconds.[14]

During the summer of 1939, Dr. Hoagland kindly brought to our laboratory his electroencephalograph, a new instrument built by Grass, with which three simultaneous records could be taken. After preliminary experimentation with the subjects sitting upright, it was decided, in order to avoid any possibility of muscle artefacts, to have the subjects in the standard position for taking electroencephalograms, namely, lying down on cots with the eyes closed. Twelve conversations between pairs of individuals were then recorded, each lasting between fifteen and twenty minutes. After a preliminary control period to obtain the brain wave record in the resting position, the subjects were instructed to talk, and the beginning of the interaction marked on the tape. At the same time, Mr. Lin recorded the beginning of the action on the recording apparatus. When the interaction ended, another control period was taken and tests were also made to eliminate the possibility that any of the muscle activities in interaction produced muscle artefacts in the record. The analysis of the records has brought out clear-cut relationships between the interaction records and brain waves not only for the delta waves but also for the alpha waves (Berger rhythm).

On each electroencephalogram tape, the values of the durations of actions and inactions for each person were plotted in the order of occurrence. This included the double actions (interruptions) and double silences (failures to respond). We thus obtain concurrent records of the electrical activity of the brain and of the interactions of the individuals.

After the record of the resting period taken as control at the beginning of

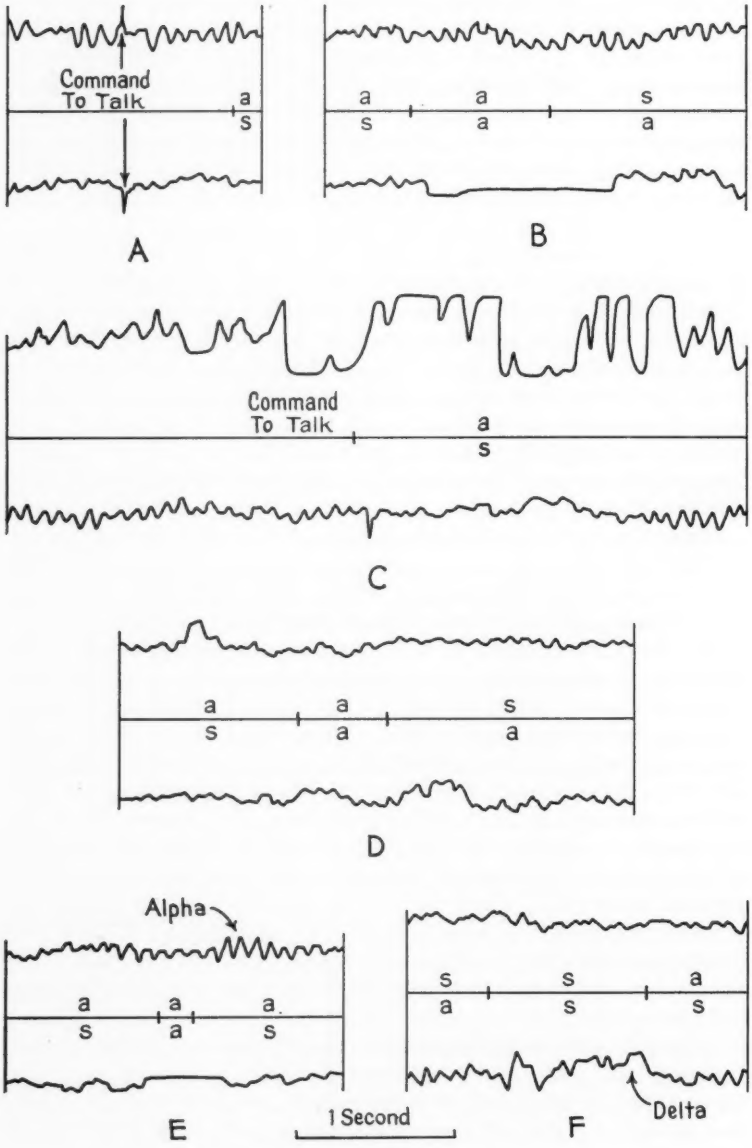
Command
To Talk
a
s
A
a a s
s a a
B
Command
To Talk
a
s
C
a a s
s a a
D
Alpha
a a a
s a s
E
1 Second
s s a
a s s
F
Delta

FIGURE 1

the observation was completed, the operator of the electroencephalograph gave the command to start talking. In all records, a characteristic "startle spike" occurred of which an example is given in *A* of figure 1. In this figure, the upper curve represents the brain wave of one individual, the lower curve the simultaneous curve of the other individual with whom he interacts. The center line has plotted on it the actions and silences of the two persons, the upper being the action-silence record of the individual whose brain wave record is given above, the lower sequence belonging to the individual having the lower brain wave. The vertical lines cutting the center line indicate the point at which a change of action takes place. The approximately similar position on the records for both individuals, and the fact that it is found in all records at the point marked in pencil on the tape where the command to begin was given, indicates that this may be regarded as an instance of a "startle pattern" in electrical activity.

(It should be explained, for the benefit of those not familar with electroencephalogram records, that alpha waves have a frequency of about ten a second and trace out fairly regular sine curves. Delta waves are slower in frequency, commonly about four a second, although they may approach about one second in their duration. Usually the term delta wave refers to all waves slower than alpha waves. In figure 1, where the scale is 3.0 centimeters to the second, examples of alpha and delta waves are pointed out.)

In double silences and double actions predominantly, and ordinarily at the beginning and ending of actions and silences, marked changes in the electroencephalograph were found. The kind of change seemed to depend upon the amount of alpha activity that the individual was manifesting on the given day. Although in a rough approximation, individuals may be classified in terms of the proportion of alpha activity on any given meter of tape during the resting period, the variation for the individual on different days or even within the same day is so great that one cannot properly speak of individuals having a high or low alpha predominance since differences of twenty-five to thirty per cent or higher occur on different days. When an individual, who at the time of observation has a per cent time alpha frequency of about fifty per cent, interrupts or is interrupted or when a double silence occurs or when he begins or ends an action or silence, there occurs a characteristic damping of the alpha waves, an example of which is to be seen in the top record in *E* of figure 1 as well as in the bottom record of *C* after the startle spike. Less frequently, besides the damping of alphas a marked delta wave will appear. In individuals who have a low per cent time alpha, and this includes individuals who on other days manifest a high alpha frequency (above 50%), there occurs a characteristic flattening of the low irregular waves into what is almost a straight line which may often be maintained to one side of the abscissa. Examples of this phenomenon are given in the lower records of *B* and *E* in figure 1. (It is of inter-

est to point out that these two records as well as the bottom record of *C* where there is much alpha activity are taken from the same individual.) In other cases, just as for individuals having higher alpha frequencies, clear-cut delta waves appear. Two examples of the appearance of delta waves are given in the bottom records of *D* and *F* in figure 1, the first for a double action and the second for a double silence. The waves associated with a change from an action to a silence, or vice versa, are illustrated by the upper record of *D* in figure 1 as well as in the extraordinary activity in the upper record of *C*. This latter record also illustrates clearly a further uniformity which we have observed. There is always an increase in delta activity (recorded by the delta index) at the beginning of interaction. In the upper record of *C* in figure 1 the subject, in a highly excited state, had a delta index of 30 during the resting period. This jumped at the beginning of interaction, marked "talk," to an estimated 90, where, as is clear from the record, the ink writer was forced continuously to the limit of its freedom of swing. One further observation is illustrated in the lower record of *C* in figure 1. At the beginning of an action or silence, the alpha waves damp out in individuals. (It must be remembered that all individuals had their eyes closed so that visual activity was eliminated as a factor in the fading of the alphas.) In long actions and in long silences, after a period of fading, the alphas came in again. In the record just referred to, at the end of the strip shown during all of which time the individual was silent, the alphas reappear on the tape. This also happens at longer intervals for cases in which the individual is acting rather than being silent.

Not only did changes of the sort described occur in association with double actions and silences and the beginning and ending of actions and silences, but changing values of the interaction rates (to be described elsewhere) were associated with changes in the frequency of delta waves. An individual arrived at the observation room and manifested a given value of the delta index in the resting period. Throughout the interaction these values changed accompanying changes in the interaction rates. The repetition of double actions, for example, and their associated delta waves, were followed by trains of delta waves which gradually disappeared, unless further stimulation took place. Although these changes may be roughly estimated by the use of the delta index, no precise method is available for quantification, since it is impossible in all cases to know how the irregular activity in the alpha waves or the long swells of over a second are related to the delta activity. Moreover, the delta index is not convenient when used in association with measures of interaction because of the difference in the operations used in the two measurements.

Characteristic of the interaction record of any individual is the cyclical fluctuation in the duration of the actions and silences. The quantitative analysis of interaction is based upon the description of these fluctuations

and the variations that occur as a result of double actions and double silences and the changes in the phase relations of the action and silence values. The measurements are all taken in durations of seconds, while the delta index is taken in terms of the excess distance traveled by a map measure over the long waves (more than 0.10 in period) over the meter distance (33.3 seconds). Since actions and silences vary in duration, the arbitrary application of the meter provides measures not based on homogeneous values of the actions and silences whose relationships provide the basis for the analysis. This matter and a preliminary solution will be discussed in a future publication. Nevertheless, it should be pointed out that increases in delta activity, as one might expect from the discussion in this paper, measured in terms of the delta index, vary with the double actions and double silences and the relationship of the actions to the silences.

The precise relationship of brain waves and interaction can only be demonstrated when parameters derived from the quantitative analysis of each type of measurement can be shown to be functionally related. Nevertheless, the evidence indicates that changes in the electroencephalogram—damping of alphas for high per cent time alphas, flattening in low per cent time alphas, as well as the occurrence of delta waves—are associated with double actions and double silences and the beginning and ending of actions and silences. The occurrence of delta waves at these points, and the appearance of levels of delta activity following changes in interaction, suggest that "emotion," if delta activity is a satisfactory measure of these changes, is produced by changing values of the interaction of individuals. The relationship of delta activity with the neuro-humoral system is not only indicated by experiments with a hypothalamic lead and the studies on the hypothalamus, but also by simultaneous measures of changes in heart rate and blood sugar concentration and the delta index.[3] It therefore seems probable that changes in the bodily states (as the physiological evidence indicates) must be regarded as activated by the stimulus-response situation set up in interaction. In any case, care must be taken in studies of emotion to control the interaction of the individuals taking part in the experiments.

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GALACTIC AND EXTRAGALACTIC STUDIES, VII. MAGNITUDES OF FORTY CEPHEIDS IN THE LARGE MAGELLANIC CLOUD

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1. Introduction.—The Large Magellanic Cloud, with more than thirteen hundred known variable stars, offers an excellent body of homogeneous material for the study of light-curve characteristics of classical Cepheid variables. Of particular interest are the problems of the correlation of period with form of light curve and the variation of amplitude with period and absolute luminosity. The interpretation of stellar atmospheres will obviously be assisted by any reliable evidence on relations between period length and the details of light variations that result from pulsations in the interiors or in the surface layers of stars.

On the basis of data obtained for galactic Cepheids, a number of observers, especially Ludendorff,[1] Hertzsprung,[2] Lundmark[3] and Gaposchkin,[4] have discussed the appearance and location on the light curves of abrupt changes in the rate of increase or decrease of light—that is, the occurrence of humps or secondary maxima and minima. Although the number of adequate light curves has been small, and the observations not altogether homogeneous, the indication of variation of form of curve with period has been fairly consistent, at least over a considerable range of period.

We turn to the wealth of material in the Large Magellanic Cloud for a further examination of the question for two reasons in particular:

(*a*) We have here a uniform system of apparent magnitudes, suitable for many Cepheids that are near together and show a wide variety of periods.

(*b*) A large number of plates made with the 24-inch Bruce refractor are now available.

The faintness of the Cepheids in the Large Cloud, and the various photometric confusions arising from the Cloud's background of rich star fields and bright and dark nebulosity, make it difficult to obtain accurate magnitude estimates. The results in the present study are therefore not as precise as we had hoped, and our light curves must be considered as a contribution to the study of a question not yet wholly settled.

2. The Observations.—Space cannot be taken to list and describe the two hundred and ninety-seven Bruce plates that are the basis of the investigation, nor is it possible to publish the individual magnitudes, eighty-two hundred in number, of the forty variables measured. The observations can, however, be made available to any investigator desiring to use them. Mean magnitudes will be published soon in *Harvard Annals*, **90**, No. 10 (1940).

The plates used are predominantly in the homogeneous series made in the decade of 1926–1936 at Bloemfontein under the supervision of Dr. J. S. Paraskevopoulos; but some date back to 1896. The early plates were included in order that the periods might be checked and improved, and evidence sought for changes of period. By decades, the numbers of plates employed are: 1896–1906, 31; 1906–1916, 7; 1916–1926, 41; 1926–1936, 195; since 1936, 23.

In table 1 the stars are listed in order of increasing period. The positions can be obtained, if desired, from the coördinates published in *Harvard Annals*, **90**, No. 1 (1933). Periods have been carefully examined by Miss McKibben, and those of the thirty-four variables previously published in *Harvard Bulletin* 905 (1937) have been revised or determined to a higher degree of accuracy. The only revisions necessary, however, were in the last decimal places of the values as published; some of the periods had been determined only approximately. The corrections to period are given in the third column of table 1. For six of the stars the periods had not been determined heretofore.

The magnitudes given in columns five, six, seven and eight are based on smooth curves drawn through normal points of five observations each. The magnitude estimates were based on five sequences that have been thoroughly intercompared and standardized.[5] For one star, H. V. 2809, the distance from the nearest sequence may have introduced a systematic error of two-tenths of a magnitude, but no correction has been made for this possible error. The number of observations in the ninth column varies because of varied plate-centering and also because plate quality has occasionally made advisable the omission of measures on faint stars.

In the last column of table 1, the description of the field is intended to indicate possible causes of unusual scattering in the observations, unusual ranges of variation or faintness produced by obscuring nebulosity. The letters have the following meanings:

p, poor star field	*e*, on edge of obscuration
r, rich star field	*o*, in obscured area
i, average star field	*o!*, very dense obscuration
c, clear of nebulosity	*d*, faint companion star in contact

d! doubling distorts image and decreases range

3. The Period-Luminosity Relation.—From columns four and eight of table 1 we find the period-luminosity relation as defined by this limited but homogeneous collection of material (Fig. 1). The line through the plotted points is

$$\log P = 7.32 - 0.42\dot{m},$$

where $\dot{m}$ is the median apparent photographic magnitude; the relation holds in the interval between periods of 2.5 and 50 days.

The dispersion about the mean period-luminosity curve can be attributed to various causes—thickness of the Cloud, intrinsic dispersion, absorption, doubling, photometric measuring errors. This scattering, and the period-luminosity relation in general, will be discussed in some detail in another paper, involving a treatment of the periods and median magnitudes of more than four hundred classical Cepheids.

4. Amplitudes, Periods and Median Magnitudes.—In figure 2*a* the relation of period to range of variation is shown, and in figure 2*b* the relation of range to median apparent magnitude. Except for the small but unknown correction for thickness of the Cloud, this second plot could be taken as range against magnitude on the absolute scale.

Again there are outstanding residuals. The three stars H. V. 5954, H. V. 2432 and H. V. 999 (circles in both diagrams) have small observed ranges which, according to the field description in table 1, are caused by doubling—the inclusion of a companion star in the measures. There is some evidence (Fig. 1) that these stars also suffer strong absorption, for otherwise their median magnitudes should be brighter than shown by the circles on the plot of the period-luminosity relation. The three deviating low-weight points have been allowed for in drawing the mean curves, which are given by

$$\log P = 1.32\,A - 0.68$$
$$\dot{m} = -2.63\,A + 18.56$$

where A is the amplitude in magnitudes, and the relations are applicable only in the interval from 2.5 to 20 days in period and from 14.7 to 16.5 in median magnitude.

That the amplitude of a Cepheid generally varies with the period has long been known,[6] but a quantitative derivation of the dependence has been hindered heretofore by the difficulties of reducing the scattered light

curves to a uniform system. The smallest ranges shown in our material may be produced by companion stars, but some galactic Cepheids (Polaris, for example) exhibit small ranges for which this explanation is less acceptable (see Hertzsprung's discussion, *loc. cit.*).

5. Period and Form of Light Curve.—Light curves of the forty Cepheids will be published with the observations in the *Harvard Annals*. Here it will suffice to summarize our somewhat indefinite conclusions concerning the relations of light-curve forms to period lengths.

(*a*) The most distinct relation found in this material is the tendency of amplitude to vary with period (and magnitude), illustrated in figure 2.

(*b*) Although there is no unambiguous dependence of shape of curve on period, the results of Ludendorff and Hertzsprung are roughly confirmed. Only three curves show distinctly double maxima, and they are the three with periods between 9.5 and 11.3 days. Less distinct secondary maxima

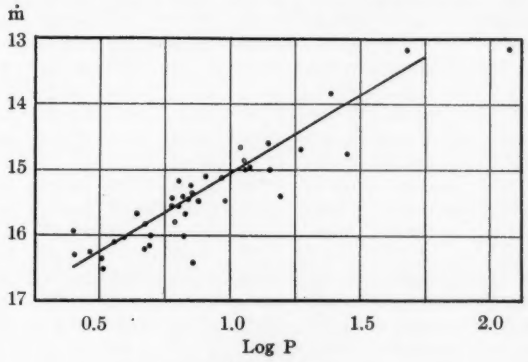

FIGURE 1
Period-luminosity relation.

or humps on the descending branch of the light curves appear throughout the series, but are most conspicuous for periods just shorter than those of the double-maximum stars. In his discussion of thirty-seven galactic Cepheids Hertzsprung finds the nearest approach to double maxima between 7.6 and 8.4 days, where our light curves are nearly smooth. This shift of the phenomenon to longer periods in the Large Cloud may be significant in Cepheid theory, if further work definitely establishes the difference.

(*c*) As also noted by Ludendorff and Hertzsprung, the light curves approach a symmetrical form when the periods are between 9.5 and 12.0 days; but the steep ascending branches are rather widely scattered throughout the whole range of periods, with highly asymmetrical light curves prevailing for all periods between 15 and 30 days.

(*d*) If we designate by a the interval of time between maximum and the moment of median magnitude on the ascending branch, and by b the interval between maximum and the moment of median magnitude on the descending branch, the ratio a/b is a measure of the asymmetry of the

TABLE 1

FORTY VARIABLES IN THE LARGE MAGELLANIC CLOUD

H. V. NO.	PERIOD IN DAYS	ΔP	LOG. PERIOD	PHOTOGRAPHIC MAGNITUDES MAX.	MIN.	RANGE	MED.	NO. OF OBS.	FIELD DESCR.
2809	2.505611	−0.000011	0.399	15.36	16.53	1.17	15.94	184	*i, c*
2344	2.524408		0.402	15.96	16.62	0.66	16.29	189	*i, e?*
2796	2.89383	−0.00003	0.461	15.79	16.72	0.93	16.26	179	*i, e?*
2368	3.2234		0.508	15.92	16.81	0.89	16.36	192	*i, e?*
2361	3.250875	+0.001125	0.512	16.03	16.98	0.95	16.51	178	*i, e*
2472	3.606651		0.557	15.71	16.49	0.78	16.10	189	*p, c?*
2795	3.913320	+0.000080	0.593	15.48	16.58	1.10	16.03	187	*i, e?*
2788	4.355729	+0.000271	0.639	15.15	16.20	1.05	15.67	192	*i, e?*
2334	4.691411		0.671	15.75	16.66	0.91	16.21	191	*i, e?*
2826	4.709949	+0.000151	0.673	15.39	16.28	0.89	15.83	193	*p, c*
951	4.8806486		0.688	15.66	16.67	1.01	16.16	186	*p, c*
2861	4.93822		0.694	15.50	16.49	0.99	16.00	198	*i, c?*
2727	5.94997	+0.00063	0.775	14.95	16.16	1.21	15.55	237	*p, o*
2619	5.97746	+0.00014	0.777	14.72	16.10	1.38	15.41	214	*p, o, d?*
5954	6.13595	−0.00595	0.788	15.51	16.07	0.56	15.79	237	*p, o, d*
2773	6.349005	−0.000005	0.803	15.03	16.08	1.05	15.55	232	*i, c?*
2536	6.37075	+0.00025	0.804	14.51	15.82	1.31	15.17	238	*i, e?*
2685	6.5415	0.0000	0.816	14.86	15.94	1.08	15.40	237	*i, e, d?*
2790	6.60615	+0.00015	0.820	15.56	16.46	0.90	16.01	185	*r, e*
2358	6.67690	−0.00690	0.825	15.32	16.02	0.70	15.67	218	*i, e, d?*
2337	6.86365	−0.00065	0.837	14.85	16.02	1.17	15.44	207	*i, e?, d?*
935	7.0674	−0.0003	0.849	14.63	15.82	1.19	15.23	249	*i, c?*
2491	7.1327	+0.0003	0.853	14.80	15.92	1.12	15.36	218	*p, c*
2752	7.1943	−0.0043	0.857	15.78	17.04	1.26	16.41	191	*p, e*
927	7.5282	−0.0002	0.877	14.87	16.07	1.20	15.47	199	*p, o!*
2722	8.02652	−0.00052	0.905	14.47	15.73	1.26	15.10	220	*i, e?*
971	9.29619	+0.00581	0.968	14.56	15.66	1.10	15.11	200	*i, e?*
952	9.5715	+0.0005	0.981	14.95	16.02	1.07	15.48	197	*i, e?*
2432	10.9245	+0.0005	1.038	14.35	14.97	0.62	14.66	201	*p, c, d*
999	11.2353	+0.0247	1.051	14.56	15.15	0.59	14.86	201	*i, c?, d!*
2787	11.442	−0.042	1.058	14.48	15.51	1.03	14.99	200	*i, e?*
905	11.85787	+0.00013	1.074	14.26	15.66	1.40	14.96	235	*r, o, d?*
2463	13.95235	−0.00235	1.145	13.80	15.41	1.61	14.60	201	*p, e?*
1006	14.21141	+0.00059	1.153	14.25	15.75	1.50	15.00	197	*r, e*
933	15.5441	+0.0009	1.192	14.66	16.15	1.49	15.40	198	*p, e*
1005	18.709759	+0.000241	1.272	13.96	15.42	1.46	14.69	199	*i, e?*
1003	24.388661	+0.006339	1.387	13.22	14.44	1.22	13.83	201	*i, c?*
934	28.18648	−0.00648	1.450	14.00	15.50	1.50	14.75	220	*i, c?*
953	47.82607	−0.12607	1.680	12.50	13.87	1.37	13.18	201	*i, c?*
2447	118.6394	−0.1394	2.074	12.65	13.66	1.01	13.16	201	*p, c?*

light curve at maximum. In groups of five stars, in order of increasing period, we have:

Mean P	$2^d.88$	$4^d.26$	$5^d.58$	$6^d.51$	$7^d.16$	$9^d.81$	$13^d.40$	$47^d.55$
Mean a/b	0.49	0.37	0.50	0.39	0.35	0.70	0.64	0.38
Av. dev.	0.09	0.05	0.10	0.07	0.08	0.34	0.19	0.13

Selecting from the sixth and seventh groups the five stars with periods between 9.5 and 12.0 days, the values are $11^d.01$, 0.88 and 0.20, showing that in this interval of period the asymmetry at maximum is least. If we

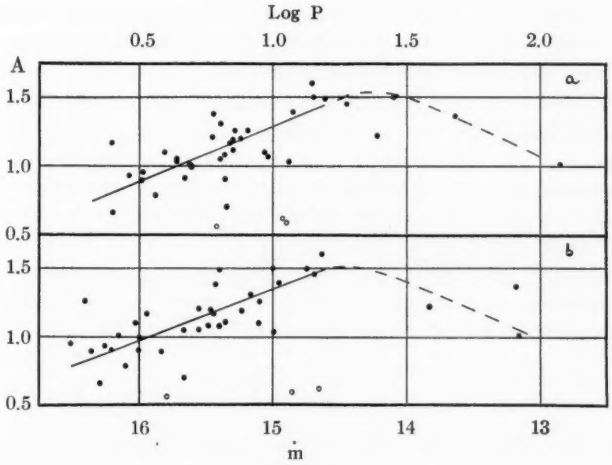

FIGURE 2

Amplitude against (a) logarithm of the period and (b) median photographic magnitude.

omit the five stars H. V. 953, 2344, 2447, 2752 and 2796, for which the scatter of the individual observations is the greatest, and for which the form of light curve may therefore be least securely determined, we have:

Mean P	$3^d.30$	$4^d.72$	$6^d.16$	$6^d.75$	$8^d.31$	$11^d.88$	$20^d.21$
Mean a/b	0.43	0.40	0.47	0.36	0.41	0.92	0.34
Av. dev.	0.07	0.07	0.12	0.03	0.09	0.17	0.08

(*e*) Evidence for a decrease, with increasing period, in the separation of maximum and hump, suggested by Ludendorff and Lundmark, is not strong in this material. At the best the shifting is a tendency, not an invariable rule; and it is not likely that by increasing the accuracy of the individual observations the exceptions will be completely removed.

6. Summary.—With what appears to be a homogeneous system of magnitude standards, the light curves of forty classical Cepheids in the Large Magellanic Cloud have been determined from estimates made on plates with the 24-inch Bruce refractor. The whole range of known periods, from 2.5 to 118 days, is represented, but special attention has been paid to stars with periods between four and twenty days.

Light elements have been found for six of the variables, and the earlier values of the periods for the other thirty-four have been revised.

Abnormally small amplitudes for three variables are attributed to the influence of close companion stars.

Throughout intervals of median magnitude and period that include most of the classical Cepheids in the Large Cloud, the amplitude increases with period and therefore with absolute magnitude.

The variation of the form of the mean light curve with the period is summarized in Section *5*.

[1] Ludendorff, *Astr. Nach.*, **209**, 217 (1919), and *Sitzb. Preus. Akad. Wiss.*, No. 5 (1929).

[2] Hertzsprung, *B. A. N.*, **2**, 83 (1924), and **3**, 115 (1926); cf. Sohon, *B. A. N.*, **3**, 204 (1926).

[3] Lundmark, *Lund Obs. Circ.*, No. 5 (1932).

[4] Gaposchkin, these PROCEEDINGS, **24**, 1 (1938).

[5] For a discussion of the magnitude system, see J. Mohr, *Harv. Ann.*, **105**, No. 11 (1937), and Shapley, *Harv. Circ.* 255 (1924).

[6] For example, see Lundmark's discussion, *Lund Medd.*, Ser. I, No. 128 (1931).

FURTHER REMARKS ON THE COSMOLOGICAL TIME SCALE

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Communicated April 5, 1940

Recent investigations on clusters of nebulae with the 18-inch Schmidt telescope on Palomar Mountain suggest that many large clusters of nebulae such as the Coma cluster represent *statistically stationary* configurations.[1] Calculations show that, if one starts from originally random distributions of nebulae, the formation of stationary large clusters requires periods of time exceeding 10^{18} years.[1] Consequently, if the process of clustering of nebulae has actually reached a stationary state, it follows that the short-time scale demanded by the hypothesis of an expanding universe is untenable.

Objections to the considerations just sketched have recently been advanced by M. S. Vallarta[2] on the ground that G. Lemaitre's theory[3] of the formation of clusters in an expanding universe was not taken into account.

The following lines are intended to show that Lemaitre's considerations in no way affect the calculations which I have advanced[1] in order to demonstrate that a stationary state of the process of clustering of nebulae can be reached only in periods longer than 10^{18} years. Indeed, the *speed* with which stationary clusters are formed has nothing whatever to do with Lemaitre's theory.* The problem of the speed of clustering is entirely of a *statistical nature* and the results obtained depend solely on the number of effective encounters among nebulae—that is, on those encounters during which appreciable amounts of the linear momenta of the involved nebulae are exchanged. Since the number and the effectiveness of such encounters depend only on the number of nebulae per unit volume, on the average velocity of these nebulae and on the law of interaction between nebulae (for which I assumed Newton's law of attraction), Lemaitre's considerations, which are also based on Newton's law of interaction between nebulae, cannot contribute anything to shorten the period necessary to achieve *statistically stationary* conditions in a large cluster of nebulae. As long as the law of interaction among nebulae is assumed to be Newton's law or any law of force not radically different from this law, a large cluster of nebulae can reach a stationary state only in a period far in excess of the time available in an expanding universe. The conclusion, therefore, remains intact, that the existence of statistically stationary configurations among clusters of nebulae is in contradiction with the hypothesis of an expanding universe.

[1] F. Zwicky, these PROCEEDINGS, **25,** 604 (1939).

[2] M. S. Vallarta, these PROCEEDINGS, **26,** 116 (1940).

[3] G. Lamaitre, these PROCEEDINGS, **20,** 12 (1934).

* Lemaitre's theory as far as I can see merely attempts to delineate the size and the material content of clusters of nebulae which may be formed in an expanding universe. It is not at all concerned with the question how long it takes for a large cluster to reach statistically stationary conditions.

THE THEORY OF THE VISUAL THRESHOLD. II. ON THE KINETICS OF ADAPTATION

By W. J. Crozier

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Communicated March 29, 1940

I. The production of a neural effect resulting in visual discrimination on the basis of brilliance requires the summation of a very large number of elements of effect produced by a large number of excitable neural units. The expression of the cumulative resultant is in terms of "elements of action" which are definable by the analytical relation of excitability to time, intensity and other experimentally controlled variables. Sufficient objective evidence already exists to support the proposition that the intensity threshold of any one excitable unit intrinsically fluctuates. So also does its contributive efficiency in the production of the elements of effect. Excitability is defined as the reciprocal of the exciting intensity; it exhibits properties justifying the conclusion that, as so defined, it is proportional to a simple velocity constant.[1] From these considerations it is deduced[2] that the relation of photic excitability $(1/\Delta I_0)$ to exposure-time must appear as a probability integral in $log\ t_{exp.}$. This is thoroughly in accord with the experimental data.[2]

II. By homologous reasoning it is deduced that threshold excitability $(1/\Delta I_0)$ should appear as a probability integral in $log\ t_D$, where t_D is elapsed time during dark adaptation. The argument, condensed, is that at any instant the units potentially excitable form a frequency distribution of $d(k)$, where k is a velocity constant governing excitability; over the finite interval required for a measurement of excitability the production of elements of effect by units in a given $d(k)$ class will decline with $t_{exp.}$, and the frequency distribution of elemental effects will then be one of $-t_{exp.}\ d(k)$, or $-t_{exp.}\ d(1/t_0)$, since k is proportional to a reciprocal time on the organism's time scale (t_0); hence, a frequency distribution of $a\ d\ log\ t$. Reasons have been given[2] for expecting that this distribution must be Gaussian. If $t_{exp.}$ is constant, the total effect obtainable is a probability integral in $log\ I$.[3] During recovery from light adaptation the frequency distribution of the excitabilities is conceived to form at any moment a frequency distribution of $d(K)$, where K is a momentary recovery velocity constant and proportional to $1/t_0$; with passage of dark-time, the number of elements in a given $d(K)$ class will decline, forming as a function of dark-time a frequency distribution in terms of $-t_D d(1/t)$, and thus of $d\ log\ t_D$. At any time t_D during dark adaptation, the excitability, measured by $1/\Delta I_0$, must then also be measured by the integral of $d\ log\ t_D$ up to $log\ t_D$. We may take this, likewise, to be Gaussian. The tests made are of two kinds: the ability of

this function to describe the data, and (more significant) the rational behavior of its three parameters when definite experimental conditions are modified.

The examination of this situation supplies a proof of the existence of random fluctuations in the individual excitabilities of elements of neural effect. Under various conditions of intensity and exposure time for light adaptation, the dark adapting elements of effect defined by $d(1/\Delta I)/d \log t_D$

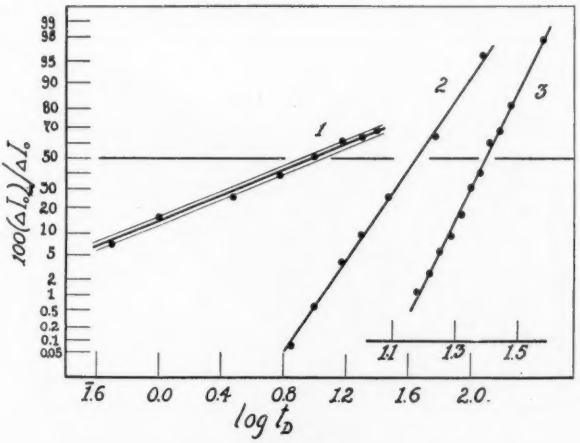

FIGURE 1

Dark adaptation contours from several sources, put on a probability grid. The abscissa is log (dark-time, minutes); the ordinate is $1/\Delta I_0$ as *per cent* of the maximum; ΔI_0 is the threshold intensity.

1—Foveal dark adaptation (red light); Kohlrausch.[7]

2—"Rod" dark adaptation; Dieter.[8]

3—"Rod" dark adaptation; Hecht, Haig and Chase[9] ("C. H.") violet light; after light adaptation to 400,000 *ml.*

The "cone" data are of course relatively more magnified; for *1*, $(\Delta I_0)_{min.}$ is taken as 4615 μL; for *2*, $(\Delta I_0)_{min.} = 5.755$ *lux;* for *3*, *log* $(\Delta I_0)_{min.} = 2.76$.

do in fact form a complete symmetrical frequency distribution as a function of *log* t_D. Moreover, the examination[4] of measurements of the retinal electrical response R shows that R is a probability integral in *log* I for different dark-times. This is impossible unless the individual thresholds fluctuate at random.

It has been shown that when area of test-spot is enlarged[3] the S. D. of the function relating threshold intensity to exposure time *increases* for the "cone" elements but *decreases* for the "rod" elements. Similar contrasting

changes occur with the S. D. of the dark adaptation function.[3] This supplies a means of showing that dark adaptation is not a matter of increasing numbers of neural units available, but depends on the increasing potential production of elements of effect.

III. The visual threshold ΔI_0 follows the probability integral law as a function of dark-time in all available series of measurements with man and all other animals that have been tested. For these measurements, in general, no quantitative formulation has been available hitherto. The non-

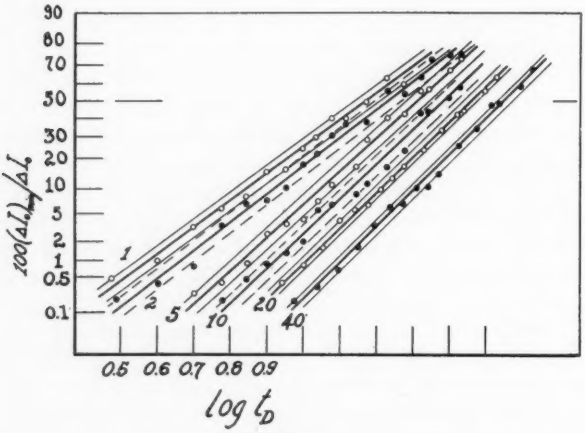

FIGURE 2

Data from Müller,[10] for dark adaptation of "rod" elements after light adaptation for 1, 2, 5, 10, 20 and 40 *min.* to 300 *lux.* The minimum value of ΔI_0 increases systematically, *log* $(\Delta I_0)_{min.}$ being (in the same order): 0.18, 0.20, 0.20, 0.24, 0.28, 0.32. The value of $\sigma'_{\log t_D}$ decreases in this order. Note also with increase of light-time the decrease in the width of the band measuring the statistically constant value of the relative scatter of t_D for each light-time; cf. [3].

(The lines for $t_L = 20$ and 40 *min.* have been moved, respectively, 0.1 and 0.2 log units to the right for clearness.)

specific form of the rule indicates that a general fact of biological organization determines its exhibition. This is found in cellular organization and in the known spontaneous but lawful fluctuation of cellular performance.

Several examples are given in figure 1. The three parameters of the probability summation are modified by the conditions of observation, but this does not alter the analytical form of the descriptive function.

The changing forms of the dark adaptation contours under various experimental conditions are also adequately described by this equation.

Systematic modifications of one or another parameter, or of several, are found as retinal region, test-spot area, antecedent adaptation, temperature, oxygen pressure and other variables are controlled. It is illustrated in this way[5] why these 3 parameters are necessary and apparently sufficient for the description of the dark adaptation contours. At the same time, there is provided a means of precisely and conveniently characterizing the nature of the influence exerted by each of these variables.

Several instances are given in figures 2, 3 and 4. For simplicity, the data illustrated in figures 2 and 3 refer only to the "rod" portions of the dark

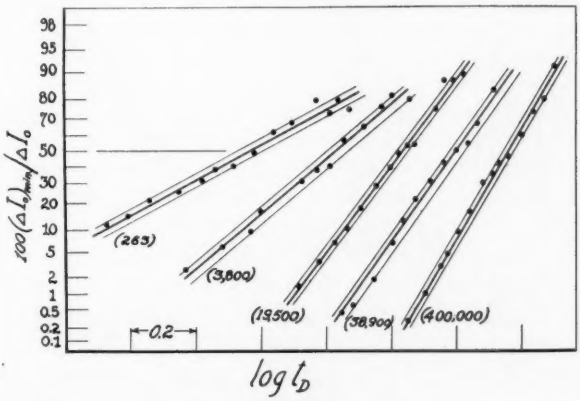

FIGURE 3

"Rod" dark adaptation after light adaptation to different intensities (photons) for a fixed time (2 *min.*), from Hecht, Haig and Chase;[9] averaged monocular determinations of ΔI_0 with violet light (data on S. H.). The values for *log* $(\Delta I_0)_{min.}$; from left to right, are: 2.40; 2.37; 2.53; 2.78; 2.74. The slope (i.e., $1/\sigma'$) decreases with increasing light adaptation, the abscissa of inflection increases and the horizontal breadth of the band decreases. The plots are shifted laterally for clearness.

adaptation contours. When the time of exposure to an adapting light is increased, or the intensity of the adapting light, the homogeneity of the frequency distribution of effect thresholds in terms of *log* t_D is increased, although its total size (maximum $1/\Delta I_0$) is less. Under reduced oxygen pressure[11] the population also becomes more uniform but is of decreased total size. This is directly accounted for by the elimination of the most light-sensitive, and consequently the most slowly adapting, elements as a result of the lowering of oxygen pressure.

The action of reduction in vitamin-*A* intake is of a slightly different character (Fig. 4). Here, the total number of available units is decreased

and its standard deviation ("rods") is decreased or ("cones") increased, but the abscissa of its inflection point changes comparatively little. The detailed examination of these matters is considered in another place.[3]

IV. The photochemical hypothesis of the visual threshold[7] postulates that its quantitative properties are determined by the receptive mechanism

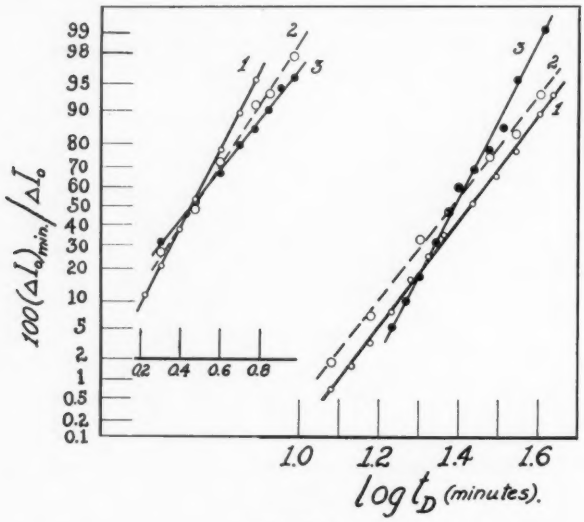

FIGURE 4

The effect on "cone" (to the right) and "rod" (to the left) dark adaptation produced by experimental restriction of dietary vitamin-*A*. These data have been published graphically by Wald, Jeghers and Arminio.[12] The lines *1* refer to the initial normal state, the points being read from a graphical analysis[3] of a number of sets of the measurements with the left eye of one subject. The lines *2* carry single observations after 15 days' restriction of vitamin-*A;* lines *3*, after 22 days. Note that the increase of slope (i.e., *decrease* of $\sigma'_{\log t_D}$) found in the "rod" data is reversed for the "cones;" this is characteristic of the "integration effect" for these 2 populations of elements,[3] and is the typical result of increasing their *number*. Here, however, it is accompanied by a decrease of total number, and points merely to a facilitation of the integration effect accompanying incipient neural injury.

at the retina, and that these properties describe the reaction-velocity kinetics of a physicochemically homogeneous system involving a photolabile substance in equilibrium with its decomposition products. The use of this hypothesis has involved further suppositions, for example that the reciprocal of the exciting intensity, or of its logarithm, is a measure of the

sensory effect produced; and basically and explicitly that the occurrence of threshold effect ("absolute" or differential) corresponds to a fixed, "constant" amount of sensory and photochemical effect. No reason is found for the acceptance of any of these assumptions, and the assumption of threshold effect constancy is contradicted.[2] The statistical theory makes none of these assumptions, provides an accurate description of the data of dark adaptation, and interprets quantitatively the changes induced by experimental treatments. This the photochemical hypothesis does not do.

The statistical theory also accounts automatically for the facts that the same analytical form of the dark adaptation contour obtains with all animals tested, and that the same kinds of changes are produced—for example, by the conditions of light adaptation—in insects and in man.

V. *Summary.* The kinetics of dark adaptation, as expressed in the change of intensity threshold with dark time, displays the statistical result of fluctuating recovery of excitability as regards the elements of neural effect produced from excited units with variable thresholds. The form of the dark adaption contour does not reveal the physiochemical nature of the metabolic process governing receptor excitation. It is simply a Gaussian probability integral in which $1/\Delta I_0$ is the ordinate and *log* (*dark-time*) is abscissa. This can be arrived at deductively. The behavior of its three parameters (maximum $[1/\Delta I_0]$; abscissa of inflection; standard deviation of the first derivative) under different experimental conditions of adaptation and of its testing, in various animals, is in accord with this.

[1] Crozier, W. J., *Proc. Nat. Acad. Sci.*, **25,** 78 (1939); Crozier, W. J., and Wolf, E., *Ibid.*, 171 (1939).

[2] Crozier, W. J., *Proc. Nat. Acad. Sci.*, **26,** 54 (1940).

[3] Crozier, W. J., "The Kinetics of Adaptation," to be published elsewhere (1940).

[4] E.g., data by Riggs, L. A., *Jour. Cell. Comp. Physiol.*, **9,** 491 (1937); analyzed in [3].

[5] Crozier, W. J., "The Kinetics of Adaptation," to be published elsewhere (1940).

[6] Hecht, S., *Physiol. Rev.*, **17,** 239 (1937).

[7] Kohlrausch, A., *Pfl. Arch. ges. Physiol.*, **196,** 113 (1922).

[8] Dieter, W., *Pfl. Arch. ges. Physiol.*, **222,** 381 (1929).

[9] Hecht, S., Haig, C., and Chase, A. M., *Jour. Gen. Physiol.*, **20,** 831 (1936–1937).

[10] Müller, H. K., *Arch. Opth.* Berlin, **125,** 624 (1931).

[11] Measurements by: McFarland, R. A., and Evans, J. H., *Amer. Jour. Physiol.*, **127,** 37 (1939); cf.[3].

[12] Wald, G., Jeghers, H., and Arminio, J., *Amer. Jour. Physiol.*, **123,** 732 (1938).

EFFECTS OF CERTAIN CHEMICAL TREATMENTS ON THE MORPHOLOGY OF SALIVARY GLAND CHROMOSOMES AND THEIR INTERPRETATION

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Communicated April 2, 1940

The difficulties met with when trying to relate the morphological structure of salivary and other chromosomes to known physico-chemical facts regarding the constituent proteins and nucleic acid molecules have suggested an attack involving the observation of chemical reactions within the chromosomes.

We subjected the salivary gland chromosomes of *Drosophila melanogaster* to treatment with solutions of varying pH ranging from 14 to 1 (i.e., 1 *N* base to 1 *N* acid). The results of these experiments, in the light of what is known about the behavior of proteins and nucleic acids toward acid and alkali, have enabled us to make some suggestions regarding the morphological as well as chemical structure of the salivary gland chromosome.

We shall present first the experiments (Kodani and Calvin), then the morphological interpretation (Goldschmidt) and finally the chemical interpretation (Calvin).

1. Experimental.—A series of buffered NaOH solutions were made up between pH 9.2 and pH 14. The solutions of pH 9.2, 9.6, 10.0 and 10.4 were borate buffers; those of pH 11.0, 11.4 and 12.0 were phosphate buffers; those of pH 12.4, 12.7 and 12.9 were glycine-NaCl buffers prepared according to Sørenson's method;[1] and those of pH above 13.0 made by diluting a normal solution of NaOH. The salivary glands were dissected from larvae in Ringer solution and then transferred to a drop of solution of given pH on a slide and treated for various lengths of time. The glands were then stained with aceto-carmine for several minutes and smeared on the slide. The time of treatment varied from 1 to 150 minutes. About 700 glands were studied.

The first observable change in structure of chromosomes after treatment is a longitudinal condensation and slight swelling which occur simultaneously so that the chromatic bands come closer to each other and break up into discrete dots (Fig. 2, *A* and *A*1). Thus the chromosomes lose their characteristic striations entirely, although they still retain a more or less clear outline. The time required to produce this structure is different for different pH's. The relative times and concentrations necessary to initiate this change are shown in figure 1. It should be noted that various points on the curve indicate merely the average times for corresponding pH's.

The area above the curve represents the region of chromosome change.

Within this area it may be noticed that the chromosomes undergo a definite sequence of changes as the time of treatment is increased. It is also found that the type of sequence is different in three different ranges of pH, namely, above 13.1, between 13.0 and 12.6 and below 12.6. The points of division into these pH ranges are not critical but represent simply the range in which there appears a definite preponderance of one type of sequence over the others. Each sequence is essentially the same for all pH's within each range, and each sequence can be divided into stages.

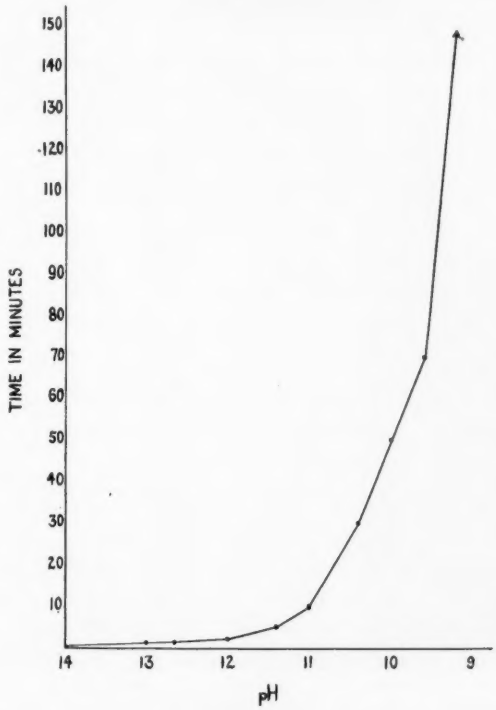

FIGURE 1

The curve indicates the relation between time and pH required to produce initial change.

The initial change described above is common to all three ranges of pH. Within the highest range of pH (above 13) this is succeeded by the following three stages. The first of these may be called the lampbrush stage because of its striking similarity to the situation in lampbrush chromosomes. At the beginning of this stage the chromosome typically shows two longitudinal

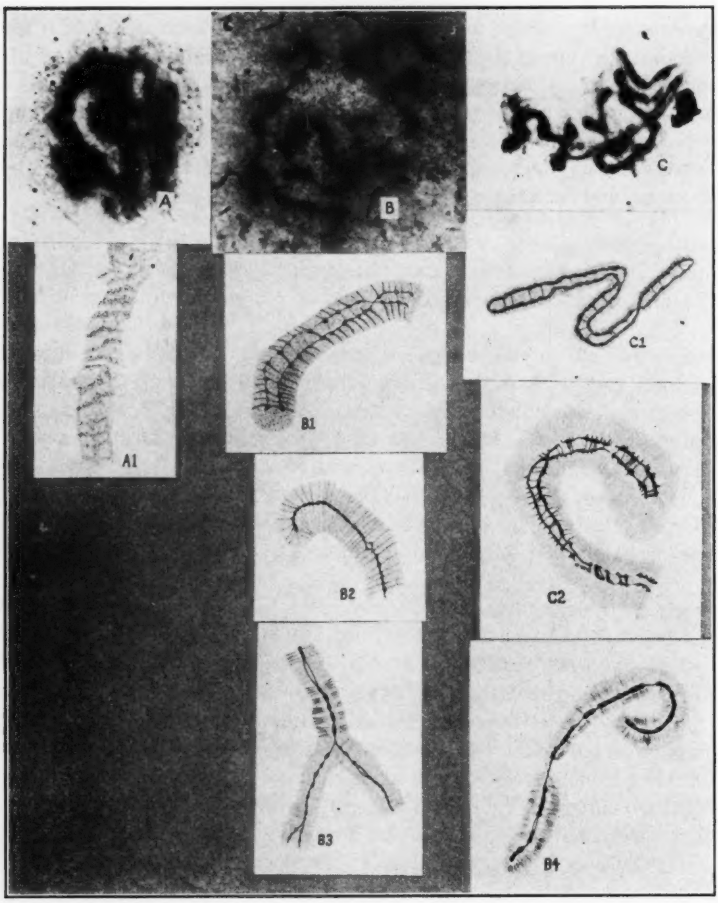

FIGURE 2

A, photomicrograph of chromosomes in Stage 1. *A*1, camera-lucida drawing of a part of chromosome in the same stage. *B*, photomicrograph of typical lampbrush chromosomes. *B*1, camera-lucida drawing of the chromosome structure at very early lampbrush stage. *B*2, camera-lucida drawing of the structure subsequent to *B*1. *B*3, camera-lucida drawing of a part of a chromosome showing differential contraction. *B*4, camera-lucida drawing of lampbrush structure from photomicrograph *B*. *C*, photomicrograph of the ladder stage. *C*1, camera-lucida drawing to show details of the early ladder stage. *C*2, camera-lucida drawing of a part of a chromosome in the late ladder stage. × ca. 1350 for all camera-lucida drawings and × ca. 1000 for all photomicrographs.

beaded strings (chromatids), paired throughout their length, with each pair of homologous chromomeres connected by a thin, sometimes beaded, transverse thread, which extends on either side to the margin of the chromosome (Fig. 2, *B*1). Subsequently the chromatids synapse, with the lateral threads still remaining in the same position (Fig. 2, *B*2). Although these steps may be seen in the same preparation, the proper sequence is indicated by relative frequencies of occurrence with respect to time and by the synapsed chromatids of the succeeding steps. The chromosome now begins to contract longitudinally. This process is suggested by the appearance in different parts of the chromatid of groups of larger chromomeres, from each of which there extend a number of lateral threads (Fig. 2, *B*3). Each of these larger chromomeres is composed of a number of smaller chromomeres, as indicated by the direct correlation between size of chromomeres and number of lateral threads. The longitudinal contraction of the chromatid proceeds irregularly at first but finally assumes the uniformly shortened appearance indicated in figure 2, *B*4 and *B*. This typical lampbrush structure marks the end of the stage. It should be noted that although only four strands are shown in figure 2, *B*3, the presence of a quadripartite structure may be seen very early in the lampbrush stage and may be considered to be present throughout.

In the next stage the chromosomes lose their lampbrush structure and finally become invisible. Occasionally, however, the chromosomes retain the outline in faint red color after the chromatic strings disappear, but after prolonged treatment these also dissolve without further change in structure.

Within the middle range of pH (13.0–12.7) the following three stages are observed. (1) Same as stage 1 in the previous range of pH. (2) At the beginning of this stage the chromosome is characterized by two thick strings which are connected by a number of fine bridges along their length, with numerous short hairs, which may be absent in some preparations, extending from the strings to the outer side. This structure of the chromonemata is retained until the next stage sets in, except that toward the end of this stage the hairs often become very much extended (Fig. 2, *C* and *C*2). This stage may be called the ladder stage because of its ladderlike structure. (3) The chromosome loses the ladderlike structure and finally becomes indistinct.

Within the lowest range of pH (below 12.6), the chromosomes go through the following two stages only. (1) The same as stage 1 of the previous ranges of pH. (2) The chromosomes disappear in the nucleus without further change in structure. In some chromosomes, however, it is occasionally observed that the achromatic regions dissolve without previous condensation or swelling, so that the chromosomes fall apart into bands and granules.

The time required to initiate various stages varies with pH, and for all three ranges the higher the pH the earlier the stages begin. Within the highest range of pH the lampbush stage begins within five minutes for all pH's and ends within about two hours and a half. In the middle range the ladder stage starts ten to thirty minutes after the beginning of the treatment and seems to persist about the same length of time as the lampbrush stage. The times stated here were determined at room temperature. Change of temperature considerably alters the occurrence and persistence of stages. A detailed investigation of this problem is now being made.

The presence of nucleic acid in the regions which take acetocarmine stain in treated chromosomes was confirmed by staining with Feulgen-Rossenbeck reagent. The existence of protein in treated chromosomes is shown by the fact that they are stained with ninhydrin. The exact distribution of the proteins at various stages of the treated chromosomes is now being studied. A number of preparations stained with Sudan IV indicated the presence of small amounts of lipoid in the native chromosomes, which were removed immediately on contact with alkali.

There seems to be no specific effect of the various ions used in the several buffers. Treatment with 1 *N* NaCl solution produced no changes in structure comparable to those of the alkali. There is, however, a general shrinkage of the chromosome after prolonged treatment with this concentrated salt solution, without any definite change in the relative orientation of the various parts. This shrinkage seems to be due simply to the dehydration resulting from the hypertonic character of the salt solution. On the other hand, normal HCl produced no visible changes even after prolonged treatment. Alkalis, such as KOH and NH_4OH, were found to produce the same effect as that of NaOH, indicating that it is the OH ion that produces the effect.

2. Morphological Interpretation.—How can these facts be interpreted morphologically in connection with known facts? Two major possibilities become visible: either the lampbrush structure is not preformed in the salivary chromosome but is an artefact produced by the treatment, or the structure is present in the salivary chromosome and made visible by the treatment. A comparison with the typical lampbrush chromosomes of the ovocytes of lower vertebrates (Rückert,[2] Carnoy-Lebrun,[3] Maréchal,[4] Koltzoff[5]) strongly suggests the second assumption. It is, moreover, made practically a certainty by Painter's[6] discovery, published while this paper was being prepared for the press, that in nurse-cell nuclei of Diptera a lampbrush structure of the chromosomes is present normally. As all the figures of the lampbrush chromosomes indicate, they consist of a central single or double chromonema with chromomeres, which, in ovocytes, is undoubtedly the stretched prophase chromosome, which later contracts into the metaphase chromosome of the meiotic division. The bristles of

the lampbrush may appear as loops attached to the chromomeres and, if covered with chromatin, may even look like transverse discs (see especially

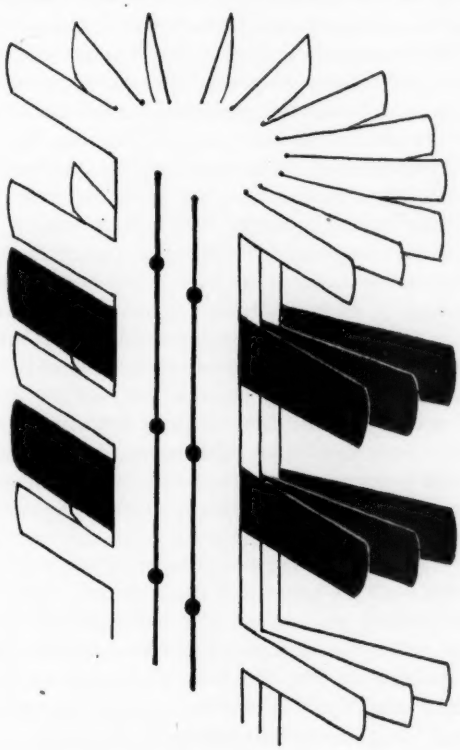

FIGURE 3

Diagrammatic representation of the arrangement of chromonemata and nuclein in the salivary chromosomes. With minor changes the diagram applies also to lampbrush chromosomes of ovocytes. One-half chromosome represented; the two main chromatids in thick lines; part of the bundle of other chromonemata in thin lines; nuclein, dark without referring to detailed structure. Two possibilities discussed in text are shown on right and left side of the diagram.

Carnoy-Lebrun and Koltzoff). The interpretation of Rückert, later elaborated by Goldschmidt,[7] was that in this condition trophochromatin is separated from idiochromatin and sloughed off into the nucleus, to be used

as material for the growth of the egg. This old interpretation was elaborated in a more modern way by Koltzoff, who considers the sloughed-off loops as replicas of the chromonema. The recent knowledge regarding intranuclear multiplication of the chromatids by division without nuclear division as visible in salivary and other gland nuclei (Geitler, a.o.[8]) suggested a similar interpretation for the lampbrush chromosomes, namely, a multiple division of the stretched chromatid into a bundle of chromonemata of which always one (or two) remain intact as the future meiotic chromosomes (idiochromatin) and retain their not completely stretched condition, whereas all others stretch completely, therefore form loops and are sloughed off as trophochromatin. This slight modification of Koltzoff's description was used in the lectures of one of the authors. The recent paper by Painter contains the part of this interpretation which relates the lampbrush to multiple division of the chromonemata, which he assumes to lead to a storage of nucleinic material in the cytoplasm to be used during development for the synthesis of chromosome material. (Painter points here to the physiologically identical but morphologically different behavior in those cases in which eggs grow by assimilation of nurse cells. Much important material pointing in the same direction can be found in Goldschmidt's old papers on trophochromatin and idiochromatin,[7] as well as viewpoints which deserve reinvestigation, starting with our present knowledge.)

Confronting, now, the facts regarding the morphology of the lampbrush chromosomes with both the experimental and morphological facts reported in the first part of this paper, as well as with Painter's recent discoveries, we come to a picture of the morphology of the salivary chromosomes as represented in the diagram (Fig. 3). The double chromonema (four in the combined salivary chromosomes), which represents the actual chromatid, is present but not visible in the untreated salivary chromosome on account of the distribution of the chromatic material (the discs). It becomes visible only when the action of the alkali concentrates the nucleic acid upon these primary chromonemata (Fig. 2). The other chromonemata form the loops (the bristles of the lampbrush), being stretched to full length and therefore coiling into loops between the serial points of attraction to the main chromonema, presumably its chromomeres. These bristles or loops are invisible in the normal condition in the salivaries (though visible in the nurse cells, according to Painter) because of their incrustation with nuclein. (Again, as in the lampbrush chromosomes of the ovocytes, the main chromonemata represent the idiochromatic part of the chromosome and the unfolded lampbrush chromonemata the trophochromatic part, functioning in the special function of the gland cell.)

One point, however, cannot be decided yet, i.e., the details of the arrangement of the nuclein (nucleic acid) in the form of dark bands. The

facts, as represented in the figures, suggest that the loops are attached to the chromomeres of the central chromonema and therefore represent the core of the chromatic discs. But it is possible that this picture is a result of the contraction after treatment and that in the natural condition the chromatic discs would be found between the loops, as indicated as an alternative possibility on the left side of the diagram. This latter assumption could be brought in line more easily with some morphological observations (Bauer[9]) as well as with the polariscopic facts (Schmidt[10]). Further experiments might furnish the decision which is hardly to be reached by pure observation. It might finally be pointed out for future use that the individual loops of the lampbrush might represent individual polypeptid chains (see also Koltzoff). In this case their order of magnitude would be 10–30,000 A and the whole chromonema would be a superchain of ca. 1000 such chains.

3. Chemical Interpretation.—Keeping in mind the spectroscopic and enzymatic experiments of Caspersson[11] and his co-workers, the optical experiments of Schmidt,[10] the stretching experiments of Buch[12] and the enzymatic experiments of Mazia and Jaeger,[13] we can extend somewhat the suggestions based on the x-ray work of Astbury and Bell[14] regarding the molecular organization of the chromonema in the light of the present results. Other models, e.g., Wrinch,[15] based upon other suggestions of protein structure, have been discussed by Gulick[16] and are rejected, since they cannot be brought into accordance with all the various facts just noted.

The essential facts derived from the present experiments are the following: (*a*) the sensitivity of the chromosome to alkali, together with its comparative stability toward salt and acid; (*b*) the sharp decrease in the rate of disintegration as the pH falls below 10; (*c*) the initial breaking up of the transverse nucleic acid bands into discrete spots; (*d*) the subsequent condensation of the nucleic acid into longitudinal strings (four strings in each chromosome pair; (*e*) the appearance of a lampbrush-like structure of the protein around the longitudinal strings; (*f*) the occasional appearance at certain pH's of a comparatively undisturbed chromosome skeleton containing no nucleic acid. (Compare with Mazia and Jaeger.)

Following Astbury and Bell, the most important elements in the molecular organization of protein fibres are the partially folded and partially extended polypeptid chain, together with the fitting of the nucleic acid onto the extended portion of the chain. These chains are held together by at least two different kinds of bonds. On the one hand there must be large groups of chains held together in a bundle by essentially primary valence bonds through the various amino acid side groups of the polypeptide chain; and on the other hand there must also operate between these primarily bound groups of chains, secondary weaker bonds which are very susceptible

to chemical or other agents. These secondary bonds are the ones which are released by alkali, and the fact that the pH at which this effect begins to occur corresponds quite well with the isoelectric point of nucleoprotein seems to indicate that we are here dealing with a neutralization of the guanidinine ion of the arginine residue which is so prevalent in nucleoproteins. The fact that this initial reaction of the chromosome to alkali has practically no temperature coefficient (a result of a part of our experiments not included in this paper), whereas the following reactions do, is confirmation of the suggestion that we are indeed concerned with a neutralization reaction. This also may account for the fixing property of acids. It is interesting to note in this connection the suggestion of Drawert[17] that the chromosome is actually in a very acid condition, the pH being between 3 and 5, so that, in alkaline solutions the hydrogen atom forming the bond would be neutralized and so reduce the attraction between groups of chains, and thus destroy the forces mainly responsible for the gross visible structure of the chromosome. The conditions of the experiment are much too mild to attack a peptide link or an ester. The initial contraction along the length and swelling of the chromosome, together with the breaking up of the chromatic bands (nucleic acid), is thus understood in terms of the relaxation of the forces maintaining the polypeptide chains in their relationship to one another and to the nucleic acid.

It should be kept in mind that none of the various structures herein described resulting from the treatment with alkali represent a condition of equilibrium of the chromosome or its constituents at the particular pH of the solution in which it was immersed. The contact with alkali doubtless started a number of parallel or consecutive chemical reactions which the fixation in acetic acid stopped at some particular point. The resulting observable condition is thus the product of several reactions whose rates most likely have quite different dependencies upon pH as well as temperature, so that it is not necessary that the identical sequence of changes be observable at different pH's differing only in the rate of their appearance. The unraveling of the various possible reactions, for example, the hydrolysis of the esters as well as the breaking of hydrogen bonds or salt links, awaits further experiment.

The next major stage in the sequence of changes is the condensation of the nucleic acid into dense longitudinal strings. This tendency of flat aromatic molecules to associate with each other, plane upon plane into long threads, is characteristic and has been observed for aqueous solutions of sodium thymonucleate as well as larger flat molecules.[18, 19] This takes place after the restraining forces have been released and the nucleic acid is free to move, either remaining attached to a loosened polypeptide chain, or entirely free of it. The fact that this condensation of nucleic acid always results in two longitudinal strings for each chromosome pair

may mean, as has been suggested in the morphological discussion, that these two strings have a primary independent existence in the untreated chromosome, or that there are simply two types of polypeptide chains together with their corresponding acids, and that there is no mixing of the two in the condensation process.

It is hoped that these and many other questions will receive an answer as further chemical experiments of this type are performed.

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ABELIAN GROUPS WHICH CONTAIN NO MORE THAN 25 PROPER SUBGROUPS

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If a group G is not the identity it contains at least two subgroups, viz., the identity and the entire group. These two subgroups are said to be improper subgroups of G while each of its other subgroups, when the order of G exceeds 2, is called a proper subgroup of G. The identity is known as an *actual* or *real* subgroup of G so that G has always one more actual subgroup than its proper subgroups, whenever G is not the identity. When G is abelian and has an order which is not a power of a prime number it is the direct product of its Sylow subgroups and the number of its subgroups is the product of the numbers of the subgroups of its Sylow subgroups. If G is the cyclic group of order p^m, p being an arbitrary prime number, then it contains exactly $m - 1$ proper subgroups. The following elementary theorem is frequently useful in the determination of the number of the proper subgroups of a given group. *A necessary and sufficient condition that there exists at least one cyclic group whose order is not a power of a prime number which contains k but no more than k proper subgroups, where k is an arbitrary positive integer, is that $k + 2$ is not a prime number.*

From the preceding paragraph it results that in order to determine the number of the abelian groups which separately involve a given number k of proper subgroups we may first find the prime power abelian groups which contain exactly k proper subgroups, then find the different possible sets of positive integral factors of $k + 2$ and determine the number of prime power abelian groups for each of these sets of factors. Hence we shall first determine the numbers of the subgroups of prime power abelian groups which separately involve no more than 25 proper subgroups. The simplest case is the one in which the abelian group is of order p^m and of the type 1^m since in this case there is a well-known formula for the number of the subgroups. When $m > 3$ the number of the distinct subgroups exceeds 25 for every value of the prime number p. This is also true when $m = 3$ except when $p = 2$. In this special case it is known that G contains 14 proper subgroups. When $m = 2$ there are $p + 1$ proper subgroups in G and for our purpose it may be assumed that $p < 29$. The number of the subgroups as p varies from 2 to 23 is as follows: 3, 4, 6, 8, 12, 14, 18, 20, 24.

When G is of type k, 1 the number of the proper subgroups of G is $kp + k$, where it may be assumed that $1 < k < 9$. As k varies from 2 to 8, and p is restricted to values so as to make the total number of proper subgroups less than 26, the number of proper subgroups is as follows: 6, 8, 12, 16, 24;

9, 12, 18, 24; 12, 16, 24; 15, 20; 18, 24; 21; 24. When G is of type k, 2 the value of k is either 2 or 3 since otherwise there would be more than 25 proper subgroups. When $k = 3$ then $p = 2$ and there are 20 proper subgroups in G. When $k = 2$ there are $p + 1$ subgroups of order p, $p^2 + p + 1$ of order p^2, and $p + 1$ of order p^3, making a total of $p^2 + 3p + 3$ subgroups. This exceeds 25 whenever $p > 3$, and is 13 or 21 as $p = 2$ or 3, respectively. The only case that remains to be considered is when $p = 2$ and G is of type 2, 1, 1. In this case there are 25 proper subgroups in G since there are 7 of order 2, 11 of order 4 and 7 of order 8.

By means of these results and the theorem stated at the close of the first paragraph of this article it is easy to determine all the abelian groups which separately contain no more than 25 proper subgroups. When the number of these subgroups is less than 15 the corresponding abelian groups are included in the lists of all the possible groups which involve a given small number of proper subgroups, which were published by the present writer in recent numbers of these PROCEEDINGS, beginning with volume 25, page 367 (1939). If an abelian group contains exactly 15 proper subgroups it follows from the preceding paragraphs that it is either the abelian group of order 64 and of type 5, 1 or the cyclic group of order p^{16}, p being any prime number. From the lists noted above it results that there are exactly three prime power abelian groups which separately contain exactly 16 proper subgroups. Besides the cyclic group of order p^{17}, where p is an arbitrary prime number, these are the abelian group of order 7^3 and of type 2, 1 and the abelian group of order 3^5 and of type 4, 1.

If an abelian group whose order is not a power of some prime number contains exactly 16 proper subgroups it contains 18 subgroups altogether. It may therefore be any one of the three cyclic groups whose separate orders are $p_1p_2^2p_3$(p_1, p_2 and p_3 being distinct prime numbers), $p_1p_2^8$ or $p_1^2p_2^5$. It may also be the direct product of the non-cyclic group of order 9 and the cyclic group of order p^2, where p is an arbitrary prime number with the exception of 3. Hence *there are seven abelian groups which separately contain exactly sixteen proper subgroups*. One of these is composed of a triply infinite system of individual groups, two of them are composed of doubly infinite systems, two of simply infinite systems, while two are individual groups. If an abelian group contains exactly 17 proper subgroups its order is a power of a prime number since $17 + 2$ is a prime number. It is therefore the cyclic group of order p^{18}, where p is an arbitrary prime number.

It was noted above that there are four prime power abelian groups which separately contain exactly 18 proper subgroups. These are the cyclic group of order p^{19}, where p is an arbitrary prime number, the non-cyclic group of order 17^2, the abelian group of order 5^4 and of type 3, 1, and the abelian group of order 2^7 and of type 6, 1. If an abelian group which contains exactly 18 proper subgroups has an order which is not a power of a

prime number it may be the direct product of the cyclic group of orders p_1, p_2 and p_3^4, p_1, p_2 and p_3 being distinct prime numbers. Whenever its order is divisible by three distinct prime numbers it may also be the direct product of the four group and the groups of orders p_1 and p_2, respectively, where p_1 and p_2 are arbitrary distinct odd prime numbers. When its order is divisible by only two distinct prime numbers it may be either the cyclic group of order $p_1^3p_2^4$, where p_1 and p_2 are arbitrary distinct prime numbers or the cyclic group of order $p_1p_2^9$.

It may also be the direct product of the four group and the cyclic group of order p^3, where p is any odd prime number, or the direct product of the non-cyclic group of order 49 and the group of any prime order except 7, or the group of order 27 and of type 2, 1 and the group of any prime order except 3. Hence *there are eleven distinct abelian groups which separately involve exactly eighteen proper subgroups*. One of these is a triply infinite system of groups, three are doubly infinite systems, four are simply infinite systems, while the remaining three are individual groups. If an abelian group contains exactly 19 proper subgroups and its order is a power of a prime number it is the cyclic group of order p^{20}. If its order is not a power of a prime number it is the direct product of two Sylow groups which involve 3 and 7 subgroups, respectively. It is therefore the cyclic group of order $p_1^2p_2^6$, p_1 and p_2 being distinct prime numbers.

The 4 prime power abelian groups which separately contain 20 proper subgroups are the cyclic group of order p^{20}, p being any prime number, the non-cyclic group of order 19^2, the abelian group of order 3^6 and of type 5, 1, and the abelian group of order 2^5 and of type 3, 2. If an abelian group whose order is not a power of a prime number contains 20 proper subgroups it is either the cyclic group of order $p_1p_2^{10}$, p_1 and p_2 being distinct prime numbers, or the direct product of the group of any odd prime order and the abelian group of 16 and of type 3, 1. Hence *there are six abelian groups which separately contain exactly twenty proper subgroups*. If an abelian group contains exactly 21 proper subgroups it is a prime power group and hence it is one of the following three groups: the cyclic group of order p^{22}, p being any prime number, the abelian group of order 2^8 and of type, 7, 1, or the abelian group of order 3^4 and of type 2, 2. If a prime power abelian group contains exactly 22 proper subgroups it is the cyclic group of order p^{23}, p being any prime number. If the order of such a group is not a power of a prime number then it is the direct product of prime power groups involving one of the following sets of numbers of subgroups such that the subgroups in each set have orders which are powers of distinct prime numbers: 2, 2, 2, 3; 4, 2, 3; 8, 3; 2, 2, 6; 4, 6; 2, 12.

The fourfold infinite system of cyclic groups of order $p_1p_2p_3p_4^2$, p_1, p_2, p_3, p_4 being distinct prime numbers, is composed of groups which separately contain 22 proper subgroups and will be regarded as a single group in accord

with the enumeration employed above. Similarly, the triply infinite system of cyclic groups of order $p_1p_2^2p_3^3$ is composed of groups which separately involve 22 proper subgroups. There is a doubly infinite system of cyclic groups whose order is of the form $p_1^7p_2^2$, p_1 and p_2 being distinct prime numbers which separately contain 22 proper subgroups. The direct product of the abelian group of order 8 and of type 2, 1 and the cyclic group of order p^3, p being any odd prime number, and the direct product of the non-cyclic group of order 25 and the cyclic group of order p^3, where p is any prime number besides 5, also contain 22 proper subgroups.

The triply infinite system of cyclic groups of order $p_1p_2p_3^5$, p_1, p_2 and p_3 being distinct prime numbers, is composed of groups which separately contain exactly 22 proper subgroups. This is also the case as regards the doubly infinite system composed of the direct product of the non-cyclic group of order 9 and the cyclic group of order p_1p_2, p_1 and p_2 being any distinct prime numbers except 3. The doubly infinite system of cyclic groups of order $p_1^3p_2^5$, p_1 and p_2 being any distinct prime numbers, is also composed of groups involving separately 22 proper subgroups. This is also true with respect to the simply infinite system composed of the direct product of the non-cyclic group of order 9 and the cyclic group whose order is the cube of any prime number besides 3. Finally, when G is the cyclic group of order $p_1p_2^{11}$, p_1 and p_2 being any distinct prime numbers, it contains 22 proper subgroups. *The total number of the abelian groups which separately contain exactly twenty-two proper subgroups is therefore eleven, infinite systems of groups of the same structure being regarded as individual groups.*

If an abelian group contains exactly 23 proper subgroups and has an order which is a power of a prime number it is the cyclic group of order p^{24}, p being an arbitrary prime number. If its order is not a power of a prime number it is the direct product of two Sylow subgroups which separately involve 5 subgroups. It is therefore either the cyclic group of order $p_1^4p_2^4$, p_1 and p_2 being distinct prime numbers, or the direct product of the four group and the cyclic group of order p^4, p being any odd prime number. It was noted above that there are 7 prime power groups which separately contain exactly 24 proper subgroups, viz., the cyclic group of order p^{25}, p being any prime number, the non-cyclic group of order 23^2, the abelian group of order 11^3 and of type 2, 1, the abelian group of order 7^4 and of type 3, 1, the abelian group of order 5^5 and of type 4, 1, the abelian group of order 3^7 and of type 6, 1, and the abelian group of order 2^9 and of type 8, 1. If an abelian group contains 24 proper subgroups and has an order which is not a power of a prime number it is the direct product of two Sylow groups which involve 2 and 13 subgroups, respectively. Hence it is the cyclic group of order $p_1p_2^{12}$, p_1 and p_2 being distinct prime numbers.

It remains only to consider the abelian groups which contain exactly 25 proper subgroups. It was noted above that there are exactly two prime

power groups which have this property, viz., the cyclic group of order p^{26}, p being any prime number, and the group of order 16 and of type 2, 1, 1. If an abelian group whose order is not a power of a prime number contains exactly 25 proper subgroups it is the direct product either of two prime power groups which separately contain 3 or 9 subgroups or of three prime power groups each of which contains three subgroups. In the former case it is the cyclic group of order $p_1^2 p_2^8$, p_1 and p_2 being distinct prime numbers, while in the latter case it is the cyclic group of order $p_1^2 p_2^2 p_3^2$, p_1, p_2, p_3 being any three distinct prime numbers. Hence *there are four abelian groups which have the common property that each of them contains exactly twenty-five proper subgroups.*

ON THE INTERSECTIONS OF IRREDUCIBLE COMPONENTS IN THE MANIFOLD OF A DIFFERENTIAL POLYNOMIAL

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1. *Formulation of Theorem.*—Let Σ be a system of differential polynomials in $y_1, \ldots, y_n$. An essential irreducible manifold in the manifold of Σ will be called an *irreducible component* (often simply *component*) of the manifold of Σ.

Let F be a form in $y_1, \ldots, y_n$. Two components of the manifold of F may have solutions in common. We are going to present a simple sufficient condition for a solution of F to belong to only a single component.

For $i = 1, \ldots, n$, a letter y_{ij}, $(j \geqq 0)$, will be called a *letter in* F if the order of F in y_i is at least j. We prove the following theorem.

THEOREM: *Let F be a form in $y_1, \ldots, y_n$. Let*

$$\bar{y}_1, \ldots, \bar{y}_n \tag{1}$$

be a solution of F. If there is a letter y_{ij} in F such that $\partial F/\partial y_{ij}$ is not annulled by (1), the solution (1) is contained in only a single irreducible component of the manifold of F.

In particular, if F vanishes for $y_i = 0$, $i = 1, \ldots, n$, and, considered as a polynomial in the y_{ik}, contains at least one term of the first degree, the solution $y_i = 0$ belongs to only one component.

2. *Proof.*—Let F_j denoted the jth derivative of F. We know that F can be decomposed into irreducible systems by choosing a sufficiently large integer p and resolving the system

$$F, F_1, \ldots, F_p, \tag{2}$$

considered as a set of simple forms, into indecomposable systems. We shall show that, for any $p \geqq 1$, (2) yields only one essential indecomposable system which vanishes when each y_{ij} in (2) is replaced by $\bar{y}_{ij}$ as determined by (1). This will prove our theorem.

Reassigning the subscripts of the y_i if necessary, let us assume that one or more $\partial F/\partial y_{1j}$ do not vanish for (1) and let m be the greatest value of j for which vanishing does not occur. Putting the forms in (2) equal to zero, we secure a set of equations which we shall regard as equations to be solved for those $y_{1,\,m+j}$ for which $0 \leqq j \leqq p$, in terms of x and the other y_{ik} in (2).

Let ξ be a value of x at which the coefficients in F and the functions in (1) are analytic and at which $\partial F/\partial y_{1m}$ does not vanish for (1). Let $[\eta]$ represent, collectively, the values at $x = \xi$ of the $\bar{y}_{ij}$ in the solution of (2) derived from (1).

The forms in (2) vanish at the point ξ, $[\eta]$ in the space of x and the y_{ij} in (2). We shall examine, at ξ, $[\eta]$, the jacobian with respect to $y_{1m}, \ldots, y_{1,\,m+p}$ of the forms in (2). In the first row of this jacobian, which row we understand to consist of partial derivatives of F, only the first term $\partial F/\partial y_{1m}$ fails to vanish at ξ, $[\eta]$. To treat the other rows, let us imagine the forms in (2) expanded in powers of the various differences $y_{ij} - \bar{y}_{ij}$. The expansion of F will contain a term $\alpha(y_{1m} - \bar{y}_{1m})$ where α is the function of x to which $\partial F/\partial y_{1m}$ reduces for (1). By the nature of m, F_1 must contain the term $\alpha(y_{1,\,m+1} - \bar{y}_{1,\,m+1})$ and can have no term $\beta(y_{1j} - \bar{y}_{1j})$ with $j > m + 1$. Thus, in the second row of the jacobian, the value of the second element at ξ, $[\eta]$ is that of $\partial F/\partial y_{1m}$ and the elements which follow have zero values. Continuing, we find the value of the jacobian at ξ, $[\eta]$ to be the $(p + 1)$th power of the value of $\partial F/\partial y_{1m}$.

Thus, for the neighborhood of ξ, $[\eta]$, $y_{1m}, \ldots, y_{1,\,m+p}$ are determined by our equations as analytic functions $f_m, \ldots, f_{m+p}$ of x and the remaining y_{ij}. By specializing the y_{ij} in the f as functions of x, we can construct solutions of (2). Indeed, we secure in this way all solutions of (2) which, in an area contained in a small neighborhood of $x = \xi$, approximate closely to the solution of (2) derived from (1).

Some essential indecomposable system—call it Σ—in the decomposition of (2) must vanish when $y_{1m}, \ldots, y_{1,\,m+p}$ are replaced by their f. This Σ must vanish for the $\bar{y}_{ij}$. If an indecomposable system Σ^1 which Σ does not hold vanishes for the $\bar{y}_{ij}$, Σ^1 has solutions which are not in the manifold of Σ and which approximate closely to the $\bar{y}_{ij}$. Thus, by what precedes, Σ is the only indecomposable system secured from (2) which has the $\bar{y}_{ij}$ as a solution. Q. E. D.

3. *The Higher Cases.* If one allows all of the $\partial F/\partial y_{ij}$ to vanish and requires the non-vanishing of one or more partial derivatives of the second order, there is no upper bound to the number of components to which a

solution may belong. We illustrate this by an example in the single unknown y. Let

$$F = y_2^2 + \prod_{j=1}^{m} [(x + j)y_1 - y]$$

where m is any integer greater than unity. Now $(x + j)y_1 - y$ has $(x + j)y_2$ as derivative and therefore has, for every j, a manifold which is a component of F. The solution $y = 0$ belongs to every such component.

DIFFERENTIALS OF FUNCTIONS WITH ARGUMENTS AND VALUES IN TOPOLOGICAL ABELIAN GROUPS[1]

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1. Introduction.—By a topological abelian group T (t.a.g. T) we shall mean an abstract abelian group—written additively—such that (*a*) the function $x + y$ and the inverse function $-x$ are continuous functions (neighborhood continuity) of both variables x and y and of the variable x, respectively, with respect to a postulated Hausdorff topology; (*b*) given any $y \in T$ and any Hausdorff neighborhood U of $0 \in T$, there exists a "positive integer" n such that[2] $y \in nU$.

In this note we shall give brief indications of a differential calculus for functions $f(x)$ with $x \in$ t.a.g. T_1 and values in a t.a.g. T_2. Proofs and further developments will appear elsewhere.

The real number system has entered into the various differential calculi studied so far in one or more of three ways: (1) through the independent and dependent variables; (2) in the topology via a numerically valued metric or norm; (3) as a multiplicative domain. *The differential calculus announced in the present paper does not employ the real numbers, thus giving a new flavor to an ancient subject and its modern generalizations.*

2. First Order Differentials.—A first order differential $f(x_0; \delta x)$ is defined as follows. In the definition of an M-differential[3] given in the paper *LTS* interpret T_1 and T_2 to be t.a.g. and not necessarily linear topological spaces. Then replace condition 2 (*b*) by the condition

$$\epsilon(x_0, x_1, nx_2) = n\,\epsilon(x_0, x_1, x_2)$$

for all "positive integers" n, for all x_1 in some Hausdorff neighborhood of $0 \in T_1$, and for all $x_2 \in T_1$. To complete the definition add the following condition:

2 (d). There exists a Hausdorff neighborhood W of $0 \in T_1$ with respect to

which the following property holds: given a Hausdorff neighborhood V of $0 \epsilon T_2$, there exists a Hausdorff neighborhood $U(V)$ of $0 \epsilon T_1$ such that

$$\epsilon(x_0, x_1, x_2) \epsilon V \text{ for } x_1 \epsilon U(V) \text{ and } x_2 \epsilon W.$$

The following theorems have been proved.

THEOREM I. *Let $f(x)$ be a function with values in a t.a.g. T_2 and defined on a Hausdorff neighborhood of $x_0 \epsilon$ t.a.g. T_1. If $f(x)$ has a first order differential $f(x_0; \delta x)$ at $x = x_0$, then $f(x_0; \delta x)$ is unique[4] for all $\delta x \epsilon T_1$.*

COROLLARY. *If $f(x)$ has a first order differential at $x = x_0$, then $f(x)$ is continuous at $x = x_0$.*

THEOREM II. *If $f_1(x)$ and $f_2(x)$ have first order differentials at $x = x_0$, then $f_3(x) = \pm n_1 f_1(x) \pm n_2 f_2(x)$ (n_1 and n_2 positive integers) has a first order differential at $x = x_0$ given by*

$$f_3(x_0; \delta x) = \pm n_1 f_1(x_0; \delta x) \pm n_2 f_2(x_0; \delta x).$$

THEOREM III. *Let T_1, T_2, T_3 be t.a.g., not necessarily distinct, and U_{x_0} a Hausdorff neighborhood of $x_0 \epsilon T_1$. If $f(x)$ on U_{x_0} to T_2 has a first order differential at $x = x_0$ and if $\phi(y)$ on $f(U_{x_0})$ to T_3 has a first order differential at $y_0 = f(x_0)$, then $\psi(x) = \phi(f(x))$ has a first order differential at $x = x_0$ given by*

$$\psi(x_0; \delta x) = \phi(f(x_0); f(x_0; \delta x)).$$

THEOREM IV. *The property of first order differentiability of a function with arguments and values in topological abelian groups is invariant under topological isomorphisms of the topological abelian groups. In particular, the invariance[5] is maintained under a passage to equivalent Hausdorff topologies of the topological abelian groups.*

THEOREM V. *If the topological abelian groups T_1 and T_2 are linear topological spaces and if $f(x)$ has a first order differential at $x = x_0 \epsilon T_1$, then the M-differential of $f(x)$ at $x = x_0$ exists and the two differentials are equal. The validity of the converse statement is an open question.*

THEOREM VI. *If the topological abelian groups T_1 and T_2 are complete normed linear spaces (Banach spaces) and if $f(x)$ has a first order differential at $x = x_0 \epsilon T_1$, then the Fréchet differential of $f(x)$ at $x = x_0$ exists and the two differentials are equal.[5] Conversely if $f(x)$ has a Fréchet differential at $x = x_0$, then a first order differential (in our sense) of $f(x)$ at $x = x_0$ exists and the two differentials are equal.*

We remark here that if we dispense with condition 2 (c) in the definition of a first order differential, then all the above theorems except Theorem V continue to hold.

3. Second Order Differentials.—nth successive first order differentials can be defined inductively whenever the $(n-1)$st successive first order differential exists for x in a neighborhood of an element x_0. In this section,

however, we shall turn our attention to an inductive definition of nth order differentials at $x = x_0$ by assuming the existence of the $(n-1)$st order differentials and the preceding differentials merely at the element $x = x_0$. For the purposes of our brief exposition, we shall do this here only for second order differentials.

Let T_1 be a t.a.g. and T_2 a t.a.g. with 0 as the only element of finite order. Let $f(x)$ possess a first order differential $f(x_0; \delta x)$ at $x = x_0$. A function $f(x_0; \delta_1 x; \delta_2 x)$ will be called a second order differential of $f(x)$ at $x = x_0$ with increments $\delta_1 x$ and $\delta_2 x$, if

(1) $f(x_0; \delta_1 x; \delta_2 x)$ is a 2-uniform symmetric bilinear function;

(2) there exists a function $\epsilon(x_0, x_1, x_2, x_3)$ with arguments in T_1 and values in T_2 such that

(*a*) $\epsilon(x_0, 0, x_2, x_3) = 0$ for all $x_2, x_3 \in T_1$,

(*b*) $\epsilon(x_0, x_1, nx_2, mx_3) = nm\,\epsilon(x_0, x_1, x_2, x_3)$ for all "positive integers" n and m, for all x_1 in some Hausdorff neighborhood of $0 \in T_1$, and for all $x_2, x_3 \in T_1$,

(*c*) $\epsilon(x_0, x_1, x_2, x_3)$ is continuous in (x_1, x_2, x_3) at $x_1 = 0, x_2 = x_2, x_3 = x_3$ for all $x_2, x_3 \in T_1$,

(*d*) there exist neighborhoods W_1 and W_2 of $0 \in T_1$ with respect to which the following property holds: given a neighborhood V of $0 \in T_2$, there exists a neighborhood $U(V)$ of $0 \in T_1$ such that $\epsilon(x_0, x_1, x_2, x_3) \in V$ for $x_1 \in U(V), x_2 \in W_1, x_3 \in W_2$;

(3) there exists some neighborhood N of $0 \in T_1$ such that for all $\delta x \in N$, $f(x_0; \delta_1 x; \delta_2 x)$ is a second order approximation to the difference $f(x_0 + \delta x) - f(x_0)$ in the sense that $2[f(x_0 + \delta x) - f(x_0) - f(x_0; \delta x)] - f(x_0; \delta x; \delta x) = \epsilon(x_0, \delta x, \delta x, \delta x)$ for all $\delta x \in N$.

THEOREM VII. *If a second order differential $f(x_0; \delta_1 x; \delta_2 x)$ of $f(x)$ exists at $x = x_0$, then it is unique*[6] *for all $\delta_1 x, \delta_2 x \in T_1$.*

THEOREM VIII. *If $f_1(x)$ and $f_2(x)$ possess second order differentials at $x = x_0$, then the second order differential of $\pm n_1 f_1(x) \pm n_2 f_2(x)$ exists at $x = x_0$ and is given by $\pm n_1 f_1(x_0; \delta_1 x; \delta_2 x) \pm n_2 f_2(x_0; \delta_1 x; \delta_2 x)$.*

THEOREM IX. *Let T_1 be a t.a.g., and T_2 and T_3 two t.a.g. without elements of finite order other than their 0 elements, and let U_{x_0} be a Hausdorff neighborhood of $x_0 \in T_1$. If $f(x)$ on U_{x_0} to T_2 possesses a second order differential at $x = x_0$ and if $\phi(y)$ on $f(U_{x_0})$ to T_3 possesses a second order differential at $y_0 = f(x_0)$, then $\psi(x) = \phi(f(x))$ possesses a second order differential at $x = x_0$ given by the formula*

$$\psi(x_0; \delta_1 x; \delta_2 x) = \phi(f(x_0); f(x_0; \delta_1 x); f(x_0; \delta_2 x)) + \phi(f(x_0); f(x_0; \delta_1 x; \delta_2 x)).$$

COROLLARY. *The correspondent of Theorem IV for second order differentiability of a function.*

In conclusion, we wish to remark that with the aid of the concept of a

product topological space it is possible to treat total differentials of functions of several t.a.g. variables.

[1] The results on first order differentials were presented to the American Math. Society at the Pasadena meeting, Dec. 2, 1939.

[2] By nx we understand the group sum $x + x + \ldots + x$ with n summands. Similarly $-nx$ will stand for the group difference $-x - x - \ldots - x$. If $S \subset T$, then by nS we mean the set of all elements nx with $x \in S$. Clearly, condition (b) in the definition of a t.a.g. T becomes redundant whenever T is specialized to be a linear topological space with real number multipliers. It is of interest to note here that throughout the whole paper, the Hausdorff separation axiom can be replaced by the weaker Fréchet separation axiom.

[3] Michal, A. D., "Differential Calculus in Linear Topological Spaces," these PROCEEDINGS, **24**, 340–342 (1938). The abbreviation *LTS* will be used to refer to this paper. See also Michal, A. D., "General Differential Geometries and Related Topics," *Bull. Amer. Math. Soc.*, **45**, 529–563 (1939).

[4] Condition (b) in the definition of a topological abelian group is used in this paper only to prove the uniqueness of the differentials for all values of the increment. Moreover the uniqueness Theorem I continues to hold even if T_2 does not satisfy condition (b). This makes possible a treatment of differentials of set-valued functions of a t.a.g. variable. We also plan to study differentials of functions of point set variables.

[5] If T_1 and T_2 are Banach spaces, then Fréchet differentiability is invariant only under a passage to equivalent Banach topologies of T_1 and T_2 whereas our differentiability is invariant under a passage to equivalent Hausdorff topologies of T_1 and T_2.

[6] We treat briefly polynomials and their polars with t.a.g. variables and then apply them to the proof of Theorem VII and its extensions.

TRIANGULATED MANIFOLDS WHICH ARE NOT BROUWER MANIFOLDS

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The following results, which will be elaborated elsewhere, grew from a study of Brouwer's definition[1] of a manifold, in its connection with questions of differentiability, analyticity and polyhedral imbedding. We first enumerate several definitions:

(1) *Topological m-manifold:* A connected topological space which can be covered by a denumerable set of neighborhoods, each of which is an m-cell.

(2) *Triangulable manifold:* A topological manifold which can be subdivided into the cells of a complex.

(3) *Star m-manifold:* A triangulated m-manifold on which the region covered by the star of any vertex is an m-cell.

(4) *Brouwer m-manifold:*[1] A star m-manifold for which the star of each vertex can be mapped homeomorphically into a euclidean m-space, E^m, so that the image of each m-cell of the star is an m-simplex.

(5) *Differentiable [analytic] m-manifold:* A topological m-manifold which can be covered with local coördinate systems having m-cells for domains, such that the transformation between any two of the systems on the intersection of their domains is differentiable [analytic] with a non-vanishing jacobian.

A few results can be stated in partial answer to the general question whether any two of the above classes of manifolds are topologically equivalent in the sense that an arbitrary member of either class has a homeomorphic image in the other. For $m < 3$, all the classes are known to be equivalent. Whitney's work[2] shows that differentiable and analytic manifolds are topologically equivalent. In a forthcoming paper,[3] the writer shows that the classes numbered (2) to (5) are all equivalent for $m = 3$. Each of the classes, as arranged above, is topologically equivalent to a subset of each preceding class.[4] The following theorem is new.

(*A*) *For every $m > 3$, there exist star m-manifolds which are not Brouwer manifolds. This is equivalent to saying that there exists, for every $m > 3$, a star of simplexes which is an m-cell but cannot be mapped into an E^m by a homeomorphism linear on each simplex of the star.*

This result is but one of the implications of a triangulation, τ, of an $(m-1)$-simplex, s ($m = 4, 5, \ldots$), constructed by the writer in such a fashion that τ is not homeomorphic to any rectilinear triangulation of s. Other results depending on the existence of such a triangulation are stated below.

By a *polyhedral representation* of a triangulated manifold, we mean the image of the manifold in a euclidean space under a homeomorphism which sets each cell of the triangulation into correspondence with a euclidean simplex. The following theorems then hold.

(*B*) *Every triangulated m-sphere, $m < 3$, has a convex polyhedral representation in E^{m+1}. For every $m \geqq 3$, there exist triangulated m-spheres which have no convex polyhedral representations in E^{m+1}.*

(*C*) *Given a finite simplicial complex, K, let n be the smallest number such that K has a polyhedral representation in E^n. Then n is not, for every K, invariant under subdivisions. In some cases, n can be made alternately to increase and decrease during a sequence of successive subdivisions.*

A polyhedral m-manifold, P^m, in E^n is said to be in *normal position*, if there exists a continuously varying transversal $(n-m)$-plane[5] $\pi^{n-m}(p)$ as p ranges over P^m. A Brouwer m-manifold, M, is characterized by the fact that it has a polyhedral representation, P^m, in an E^n, such that transversal $(n-m)$-planes exist at each vertex. The writer has partly treated (see *H*) the question of the possibility of extending the definition of these transversal planes over the whole of P^m, thus showing it to be in normal position.

If and only if this can be done, it is possible (see *H*) to apply Whitney's method[2] of constructing an analytic approximation to P^m and hence a homeomorphic image of M in the class of analytic manifolds. Hence part of the significance of the following statement.

(*D*) *The star manifolds of Theorem* (*A*) *above do not admit polyhedral representations in normal position. For every* $m > 3$, *there exist stars of simplexes which are m-cells but do not admit transversal* $(n - m)$*-planes no matter how they are polyhedrally represented in any* E^n.

It is uncertain whether, by changing the triangulation, these same star manifolds can always be made into Brouwer manifolds. If not, then there exist manifolds which cannot be made differentiable.[4]

(*E*) *Brouwer's definition*[1] *of an m-manifold is not invariant* $(m > 3)$ *under subdivisions.*

Some of the above results have interesting interpretations in the space, Π, of $(n - m)$-planes through a point, O, in E^n. A subspace of Π is the space $\Pi(S)$ of planes transversal to a given star, S, of m-simplexes incident with O. The space Π can be defined with the aid of Vahlen's relations among the determinants of a matrix, and $\Pi(S)$ is then definable by supplementary conditions on these determinants. For the cases where S admits no transversal $(n - m)$-plane, we have the algebraic result that the system of equations and inequalities defining $\Pi(S)$ is inconsistent.

[1] L. E. J. Brouwer, "Über Abbildungen von Mannigfaltigkeiten," *Math. Ann.*, **71**, 97–115 (1912).

[2] Hassler Whitney, "Differentiable Manifolds," *Ann. Math.* **37**, 645–680 (1936).

[3] S. S. Cairns, "Homeomorphisms between Topological Manifolds and Analytic Manifolds," to appear in the *Ann. Math.* This paper is hereafter referred to as *H*.

[4] For every differentiable manifold can be triangulated into a Brouwer manifold. This follows, with the aid of Whitney's imbedding theorem (loc. cit.), from certain results obtained by the writer. See S. S. Cairns, "Polyhedral Approximations to Regular Loci," *Ann. Math.* **37**, 409–415 (1936).

[5] Such a plane is characterized by the conditions that it pass through p and make angles bounded away from zero with the secants of a neighborhood of p on P^m.